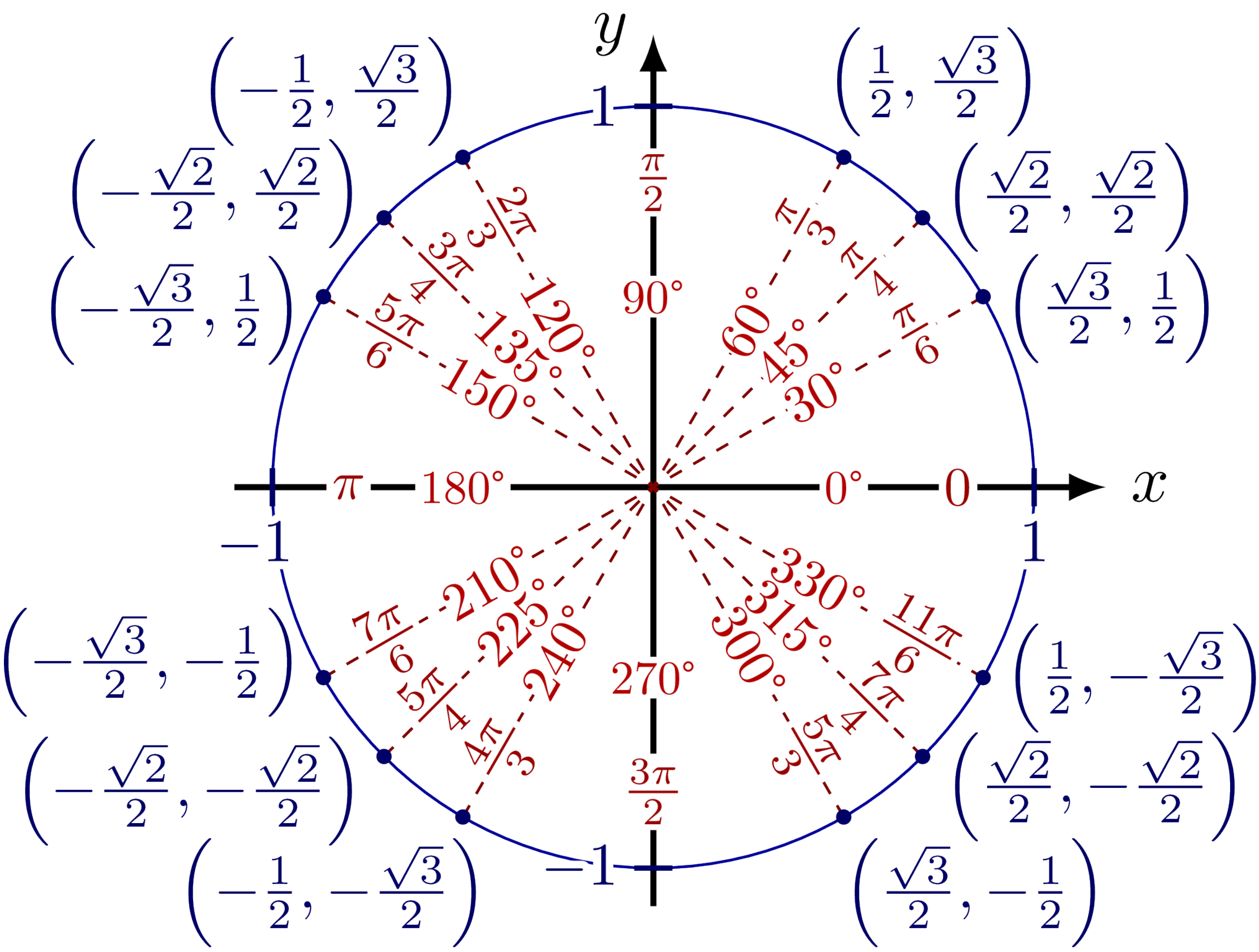
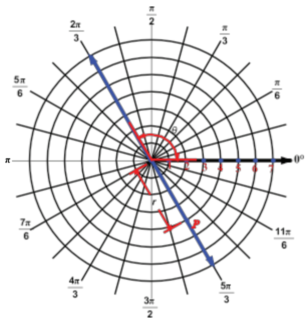
CHAPTER (17)

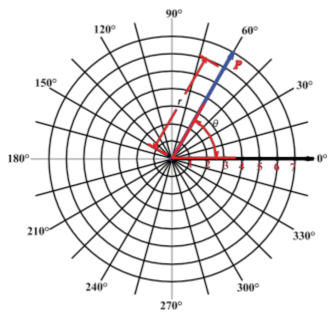
POLAR COORDINATES AND COMPLEX NUMBERS



Polar Coordinates



$P\left(-4, \dfrac{2\pi}{3}\right)$



$P(6, 60°)$

- Polar plane is used for example when air traffic controllers record the locations of airplanes using distances and angles.
- The location of a point P in the polar coordinate system can be identified by polar coordinates of the form (r, θ), where

- r represents the directed distance from the pole to the point
- θ represents the directed angle from the polar axis $\overrightarrow{OP}$
- θ is positive indicates a counter clockwise rotation from the polar axis
- θ is negative indicates a clockwise rotation
- r is negative indicates a clockwise rotation
- r is positive, then P lies on the terminal side of θ
- r is negative, then P lies on the ray opposite the terminal side of θ

- If the terminal side rotates one full circle, then it will be drawn again on the same point.

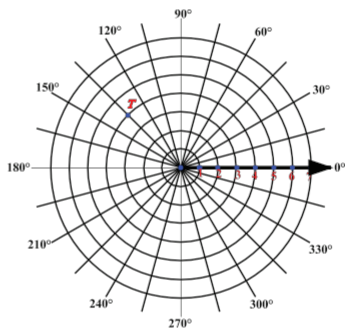
$\rightarrow (r, \theta) = (r, \theta \pm n \cdot 360)$ *or* $(r, \theta \pm 2n\pi)$

On the other hand, because r is a directed distance

$(r, \theta) \;\rightarrow\; (-r, \theta \pm m180)$ *or* $(-r, \theta \pm m\pi)$

where n is integer m is odd integer

1. **Find a different pair of polar coordinates that name the point** $T\,(4, 135)$



A $(-4, 135)$ B $(-4, 315)$

C $(4, 315)$ D $(4, -45)$

$\theta = 135°$

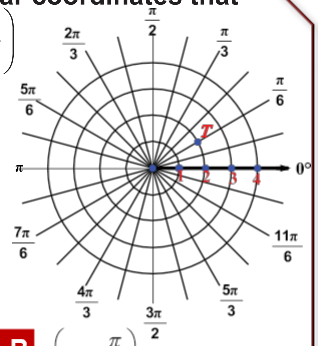
Think of $(-r, \theta \pm 180)$ or $(r, \theta \pm 360)$

$(4, 135) = (-4, 135 + 180)$

$= (-4, 315)$

➢ **B**

2. **Find a different pair of polar coordinates that name the point** $T\left(2, \dfrac{\pi}{6}\right)$



A $\left(2, \dfrac{7\pi}{6}\right)$ B $\left(-2, \dfrac{\pi}{6}\right)$

C $\left(2, \dfrac{11\pi}{6}\right)$ D $\left(2, \dfrac{-11\pi}{6}\right)$

Think of $(-r, \theta \pm \pi)$ or $(r, \theta \pm 2\pi)$

$(r, \theta - 2\pi) \;\rightarrow\; \left(2, \dfrac{\pi}{6} - 2\pi\right)$

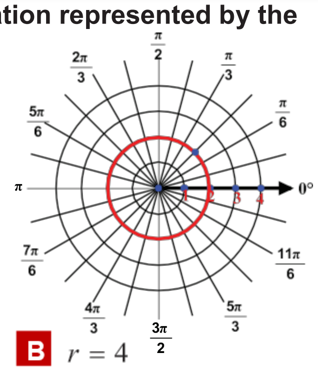
$\left(2, \dfrac{\pi}{6} - \dfrac{12\pi}{6}\right) = \left(2, \dfrac{-11\pi}{6}\right)$

➢ **D**

Graphs of Polar Equations

- An equation expressed in terms of polar coordinates is called polar equation.
- A polar graph is the set of all points with coordinates (r, θ) that satisfy the given polar equation.

3. **Find the polar equation represented by the graph**



A $r = 2$ B $r = 4$

C $r = 3$ D $r = 6$

The graph is a circle centered at the origin with radius **2**, therefore the graph consists of all points that are **2** units from pole

➢ **A**

$\rightarrow \; r = 2$

4. What is the diameter of the circle represented by the polar equation $r = 6$

A 3 **B** 6

C 9 **D** 12

The polar equation $r = 6$ represents a circle with radius $= 6$

$d = 2r$

$= 2 \quad 6$

$= 12$

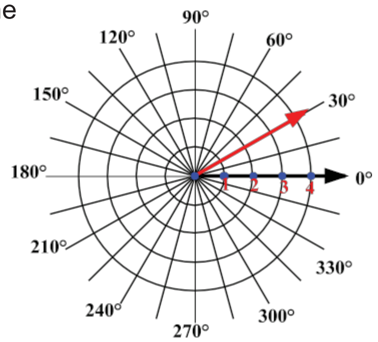
D

5. The graph of θ 30^0 in the polar plane represents a …

A Point **B** Line

C Circle **D** Radius $= 30$

By graphing $\theta = 30^0$ in the standard position then it is a line



B

Polar Distance Formula

If $P_1(r_1, \theta_1)$ *and* $P_2(r_2, \theta_2)$ are two points in the polar plane, then the distance P_1P_2 is given by

$$\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

6. Find the distance between $P_1(0, 160°)$ and $P_2(5, 140°)$

A 25 **B** 50

C 10 **D** 5

$$P_1P_2 = \sqrt{5^2 + 0^2 - 2 \cdot 0 \cdot 5 \cos(160 - 140)}$$

$$= \sqrt{25 + 0 - 0}$$

$$= 5$$

D

7. Let the distance between 13, $P_1(r, 0)$ and $P_2(12, 90^0)$ be then find r

A 5 **B** 25

C 131 **D** 10

$$P_1P_2 = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

$$= \sqrt{r_1^2 + 12^2 - 2r_1 \cdot 12 \cos(90 - 0)}$$

$$13 = \sqrt{r_1^2 + 144 - 24r \cdot 0}$$

$$13 = \sqrt{r_1^2 + 144}$$

$$\left(13 = \sqrt{r^2 + 144}\right)^2$$

$$169 = r^2 + 144$$

$$25 = r^2$$

$$r = 5$$

A

Polar and Rectangular Forms of Equations

- Convert Polar to Rectangular Coordinates.

$x = r\cos\theta, \; y = r\sin\theta$

That is $(x, y) \;\to\; (r\cos\theta, r\sin\theta)$

- Convert Rectangular to Polar Coordinates

$r = \sqrt{x^2 + y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad x > 0$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi \qquad x < 0$$

8. Find the rectangular coordinates for the point $P\left(4, \frac{\pi}{6}\right)$

A $(2, 2\sqrt{3})$ **B** $(2\sqrt{3}, 2)$

C $(2\sqrt{2}, 3)$ **D** $(2\sqrt{3}, -2)$

For $P\left(4, \frac{\pi}{6}\right)$ $r = 4$ and $\theta = \frac{\pi}{6}$

$x = r\cos\theta$
$= 4\cos\frac{\pi}{6}$
$= 4\frac{\sqrt{3}}{2}$
$= 2\sqrt{3}$

$y = r\sin\theta$
$= 4\sin\frac{\pi}{6}$
$= 4 \cdot \frac{1}{2}$
$= 2$

$\to (2\sqrt{3}, 2)$

B

9. Find the rectangular coordinates for the point

$Q(-2, 135)$

A $\left(-\sqrt{2}, \sqrt{2}\right)$ B $\left(2, \sqrt{2}\right)$

C $\left(\sqrt{2}, 2\right)$ D $\left(\sqrt{2}, -\sqrt{2}\right)$

For $Q(-2, 135)$, $r = -2$ and $\theta = 135$

$x = r\cos\theta = -2\cos 135 = -2\left(\frac{-\sqrt{2}}{2}\right) = \sqrt{2}$

$y = r\sin\theta = -2\sin 135 = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$

$\rightarrow \left(\sqrt{2}, -\sqrt{2}\right)$

D

10. Find the polar coordinates for the point with the rectangular coordinates $P\left(1, -\sqrt{3}\right)$

A $\left(2, 120^0\right)$ B $\left(2, -120^0\right)$

C $\left(2, -300^0\right)$ D $\left(2, 300^0\right)$

For $P\left(1, -\sqrt{3}\right)$ $x = 1$ $y = -\sqrt{3}$

$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{1} = 300^0$

D

11. Write the rectangular equation in polar x^2 form

A $r = \tan\theta\sec\theta$ B $r = \cot\theta\csc\theta$

C $r = \cot\theta\sin\theta$ D $r = \tan\theta\cos\theta$

$y = x^2$

$r\sin\theta = (r\cos\theta)^2$

$r\sin\theta = r^2\cos^2\theta$

$\sin\theta = r\cos^2\theta$

$r = \frac{\sin\theta}{\cos^2\theta}$

$= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$

$r = \tan\theta\ \sec\theta$

A

12. Write the rectangular equation $x^2 - y^2 = 1$ in polar form

A $r = \sqrt{\csc 2\theta}$ B $r = \sqrt{\sec 2\theta}$

C $r = \sqrt{\cot 2\theta}$ D $r = \sqrt{\tan 2\theta}$

$x^2 - y^2 = 1$

$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1$

$r^2(\cos 2\theta) = 1$

$r^2 = \frac{1}{\cos 2\theta}$

$r^2 = \sec 2\theta$

$r = \sqrt{\sec 2\theta}$

B

13. Write the rectangular equation in polar 4 form

A $2\sec\theta$ B $4\csc\theta$

C $4\sec\theta$ D $4\cot\theta$

$x = 4$

$r\cos\theta = 4$

$r = \frac{4}{\cos\theta} = 4\sec\theta$

C

Complex Numbers

- The absolute value of the complex number $z = a + bi$ is $|z| = |a + bi| = \sqrt{a^2 + b^2}$
- The polar form of the complex number $z = a + bi$ is: $z = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{a^2 + b^2}$
- $a = r\cos\theta,\ b = r\sin\theta$

  $\theta = \tan^{-1}\frac{b}{a}$ $a > 0$

  $\theta = \tan\frac{b}{a} + \pi$ $a < 0$

14. Find the absolute value of the complex number $z = -2 - i$

A 5 B $\sqrt{5}$

C 3 D $\sqrt{3}$

$|z| = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (-1)^2} = \sqrt{4 + 1} = \sqrt{5}$

B

15. **Express the complex number**

$$z = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

in rectangular form

A $3\sqrt{3} + 3i$ **B** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

C $\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ **D** $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

$$3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

➢ C

DeMoivre's Theorem

If the polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$,
then for positive integers n

$$z = \left[r\left(\cos\theta + i\sin\theta\right)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Distinct Roots

For a positive integer p, the complex number $r(\text{Cos}\,\theta + i\,\text{Sin}\,\theta)$ has p distinct pth roots. They are found by $r^{\frac{1}{p}}\left(\cos\frac{\theta + 2n\pi}{p} + i\sin\frac{\theta + 2n\pi}{p}\right)$

where $n = 0, 1, 2, \ldots.., p-1$

When n equals or exceeds p, the roots repeat as the following:

$$\frac{\theta + 2\pi p}{p} = \frac{\theta}{p} + \frac{2\pi p}{p}$$

$$= \frac{\theta}{p} + 2\pi$$

16. **Evaluate** $\left[2\left(\cos 37\cdot 5 + i\sin 37\cdot 5\right)\right]^4$

A $-8\sqrt{3} + i$ **B** $8\sqrt{3} + i$

C $-8\sqrt{3} + 8i$ **D** $8\sqrt{3} - 8i$

$$\left[2\left(\cos 37\cdot 5 + i\sin 37\cdot 5\right)\right]^4$$

$$= 2^4\left[\cos\left(4\cdot 37\cdot 5\right) + i\sin\left(4\cdot 37\cdot 5\right)\right]$$

$$= 16\left(\cos 150 + i\sin 150\right)$$

$$= 16\left(\frac{-\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

$$= -8\sqrt{3} + 8i$$

➢ C

17. **What is $\frac{\theta + 2n\pi}{p}$ for the first root of the fourth roots of the complex number**

$$4\sqrt{2}\left(\cos\frac{5\pi}{4}, i\sin\theta\frac{5\pi}{4}\right)$$

A $\frac{5\pi}{16}$ **B** $\frac{13\pi}{16}$

C $\frac{21\pi}{16}$ **D** $\frac{5\pi}{4}$

The first root $\rightarrow n = 0,$ fourth roots $\rightarrow p = 4$

$$\theta = \frac{5\pi}{4}$$

$$\frac{\theta + 2n\pi}{p} = \frac{\frac{5\pi}{4} + 2(0)\pi}{4}$$

$$= \frac{\frac{5\pi}{4}}{4}$$

$$= \frac{5\pi}{16}$$

➢ A