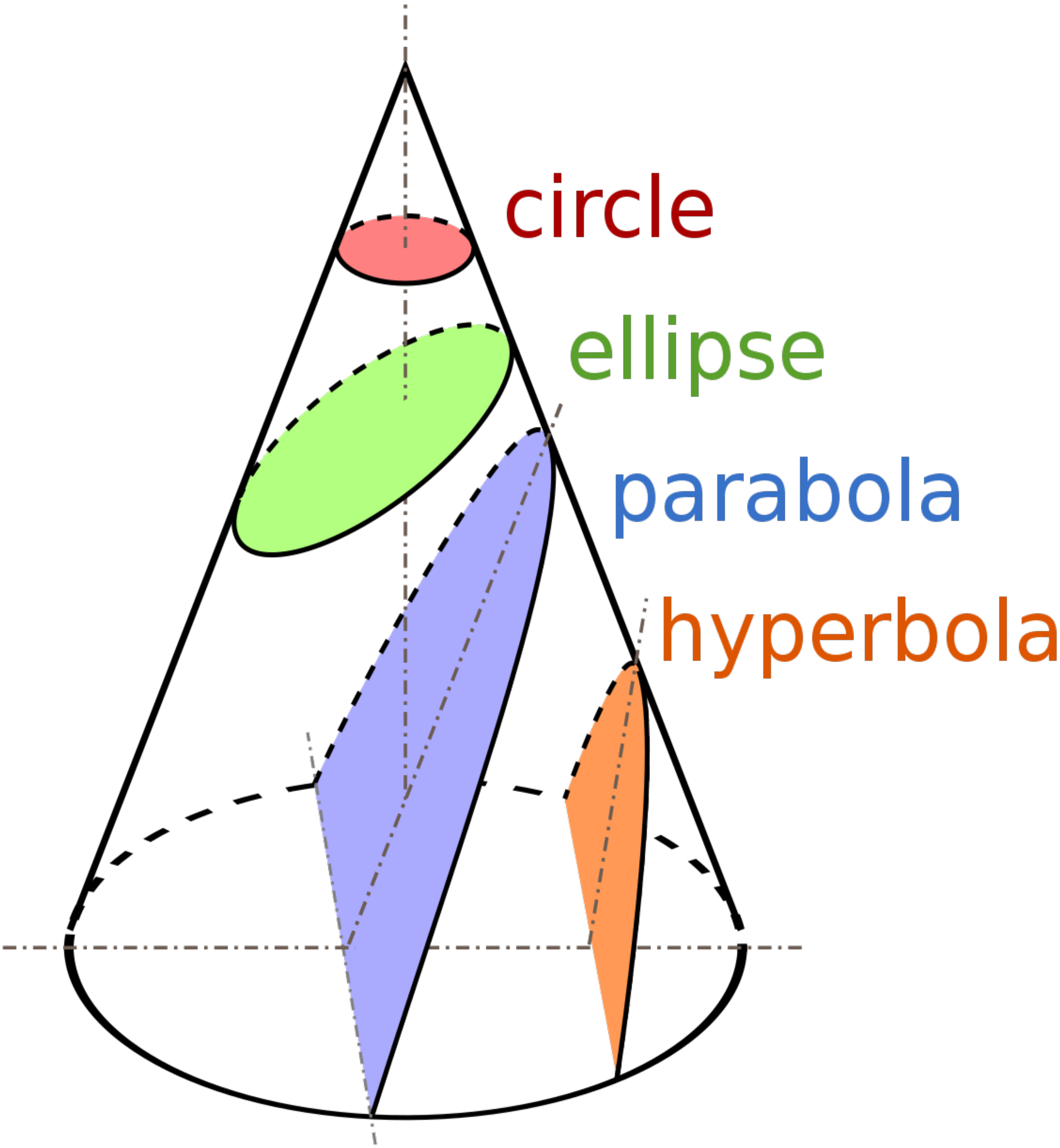
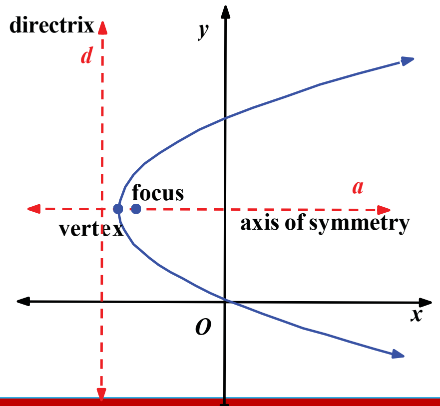
CHAPTER (15)
Conic Sections



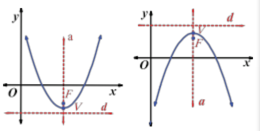
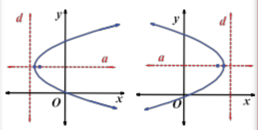
1) Parabolas

A parabola represents all points in a plane that are equidistant from a fixed point called the focus and a specific line called the directrix.

Usually, the distance from the vertex to the focus is denoted by $|p|$ or $|c|$ and it is the same distance from the vertex to the directrix. It is called: ***focal length.***



Standard Form of Equations for Parabolas

Equation	$(x-h)^2 = 4p(y-k)$	$(y-k)^2 = 4p(x-h)$
Graph		
P	$p > 0$ $p < 0$	$p > 0$ $p < 0$
Orientation	Opens vertically	Opens horizontally
Vertex	(h, k)	(h, k)
Focus	$(h, k+p)$	$(h+p, k)$
Axis of Symmetry	$x = h$	$y = k$
directrix	$y = k - p$	$x = h - p$
Focal chord	$\|4p\|$	$\|4p\|$

15.1 Find the focal length of the parabola $(y-10)^2 = 8(x-7)$

A 2 **B** 8

C 4 **D** 16

Compare to $(y-k)^2 = 4p(x-h)$ $|p|$ is the focal length

$4p = 8$

$p = 2$ ➤ **A**

15.2 Find the focal chord of the parabola $(y-10)^2 = 8(x-7)$

A 2 **B** 8

C 4 **D** 16

Compare to $(y-k)^2 = 4p(x-h)$ $|4p|$ is the focal chord

$|4p| = |8| = 8$ ➤ **B**

15.3 Find the equation of the axis of symmetry to the parabola $(y-6)^2 = -4(x+2)$

A $y = 6$ **B** $y = -6$

C $x = 2$ **D** $x = -2$

The equation is in standard form and the squared term is y, which means that the parabola opens horizontally

$\rightarrow$ Axis of symmetry $y = k$

$y = 6$ ➤ **A**

15.4 What is the orientation and direction of the parabola $x^2 = -3(y-7)$

A Vertically up **B** Vertically down

C Horizontally right **D** Horizontally left

The equation $(x-0)^2 = -3(y-7)$ is in standard form and the squared term is x, which means that the parabola opens vertically

$4p = -3$

$p = \frac{-3}{4} \rightarrow p < 0$ So it opens downward ➤ **B**

15.5 What is the orientation and direction of the parabola with a focus $(3, 2)$ and directrix $y = -2$

A Vertically up **B** Vertically down

C Horizontally right **D** Horizontally left

Since the directrix $y = -2$ is a horizontal line then the parabola opens vertically, on the other hand the focus is above the directrix then it opens upward. ➤ **A**

15.6 Find the vertex of the parabola $4(x-3) =$

A $(-3, 5)$ **B** $(3, -5)$

C $(5, -3)$ **D** $(-5, 3)$

Compare to $4p(x-h) = (y-k)^2$

$h = 3$

$k = -5$

➤ **B**

Focus is $(h, k) \rightarrow (3, -5)$

15.7 Write the equation of the parabola with the focus $(0, 0)$, axis of symmetry is the y-*axis* and the parabola passes through the point $(2, -1)$

A $4x^2 = y$ **B** $x = -4y^2$

C $x^2 = -4y$ **D** $4x = y^2$

The *axis* of symmetry is the y-axis then it is opens vertically and the squared term should be x, therefore options B and D are eliminated.

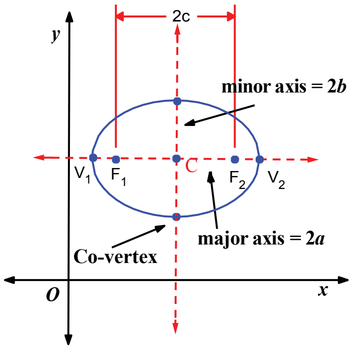
The given points should satisfy the required equation.

- $(0, 0)$ satisfy all equations.
- The point $(2, -1) \rightarrow x^2 = -4y \rightarrow (2)^2 = -4(-1)$ True

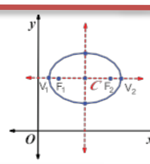
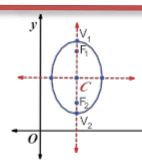
➢ **C**

2) Ellipses

An ellipse is all points in a plane such that the sum of the distance from two fixed points, called foci, is constant.



Distance and Symbols		
From	To	Symbol
Vertex	Center	a
Co-vertex	Center	b
Focus	Center	c
Vertex	Vertex	$2a$ = major axis
Co-vertex	Co-vertex	$2b$ = minor axis
Focus	Focus	$2c$

Standard Forms of Equations for Ellipses		
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Graph		
Orientation	Horizontal major axis	Vertical major axis
Center	(h, k)	(h, k)
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Co-vertices	$(h, k \pm b)$	$(h \pm b, k)$
Major *axis*	$y = k$	$x = h$
Minor *axis*	$x = h$	$y = k$
a, b, c **relationship**	$a > b$ $c^2 = a^2 - b^2 \rightarrow c = \sqrt{a^2 - b^2}$	
Eccentricity	$e = \frac{c}{a}$ $0 < e < 1$ If $e = 0$, then the ellipse becomes a circle.	

15.8 Determine the equation of the ellipse if its center is at $(4, -5)$

A $\frac{(x+5)^2}{16} + \frac{(y-4)^2}{25} = 1$ **B** $\frac{(x-4)^2}{16} + \frac{(y+5)^2}{25} = 1$

C $\frac{(x-5)^2}{25} + \frac{(y+4)^2}{16} = 1$ **D** $\frac{(x+4)^2}{25} + \frac{(y-5)^2}{16} = 1$

Center is at $(4, -5) \rightarrow h = 4$ and $k = -5$

$$\frac{(x-4)^2}{\square} + \frac{(y-(-5))^2}{\square}$$

$$\frac{(x-4)^2}{\square} + \frac{(y+5)^2}{\square}$$

➢ **B**

The denominator does not affect the center

15.9 Find the minor axis of the ellipse $\frac{(x-12)^2}{49}+\frac{(y+4)^2}{25}=1$

A 5 **B** 10
C 7 **D** 14

The minor axis is the shortest axis $2b$

$b^2=25$

$b=5$

$2b=2\times5=10$ ⋙ **B**

15.10 Find the distance from the center to the vertex of the ellipse $\frac{x^2}{9}+\frac{(y-1)^2}{16}=1$

A 8 **B** 4
C 3 **D** 6

The distance from the center to the vertex is $\boldsymbol{a}$ and to the co-vertex is $\boldsymbol{b}$

$\boldsymbol{a}$ is major (longest) axis

$a^2=16$

$a=4$ ⋙ **B**

15.11 If the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{M}=1$ and its focus is $(0,3)$ then what is the value of M?

A 9 **B** 25
C 16 **D** 10

Method 1

$\boldsymbol{h}$ and $\boldsymbol{k=0}$

Center is $(0,0)$ the focus is $(0,3)$ then the ellipse is vertical. a^2 is below y^2 and $M>16$

The only option is B

Method 2

The focus is $(0,3)\to c=3$

$b^2=16\to b=4$

$c^2=a^2-b^2$

$3^2=a^2-16$

$a^2=25=M$ ⋙ **B**

15.12 Find the length of the minor axis of the ellipse $\frac{(x-3)^2}{25}+\frac{(y+4)}{64}=1$

A 5 **B** 10
C 8 **D** 16

the minor axis is $2\boldsymbol{b}$

$b^2=25$

$b=5$

$2b=2\times5$

$=10$ ⋙ **B**

15.13 If the distance between the foci of an ellipse is 16 and the length of the minor axis is 12, then find its eccentricity.

A 6 **B** 8
C $\frac{4}{5}$ **D** $\frac{4}{3}$

The distance between the foci is $2c=16$
$c=8$

The length of the minor axis is $2b=12$
$b=6$

$c^2=a^2-b^2$

$8^2=a^2-b^2$

$64+36=a^2$

$100=a^2$

$10=a$

$e=\frac{c}{a}$

$=\frac{8}{10}$

$=\frac{4}{5}$ ⋙ **C**

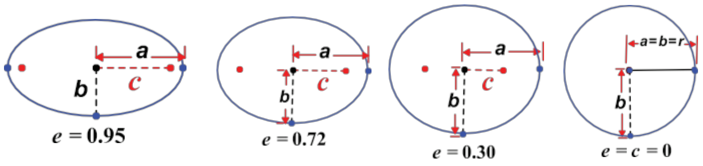
15.14 The range of the eccentricity of an ellipse is varies between 0, and

A $\frac{1}{2}$ **B** 1
C $1\frac{1}{2}$ **D** 2

Eccentricity $0<e<1$ ⋙ **B**

3) Circles

The value of c represents the distance between one of the foci and the center of the ellipse. As the foci moves closer together, c and e approach 0 the ellipse is a circle and both a and b are equal to the radius of the circle.



Standard Form of Equation for Circles

The standard form of an equation for a circle with center (h, k) and radius r is $(x-h)^2+(y-k)^2=r^2$

15.15 Find the center and the radius of the circle $(x-1)^2+(y+2)^2=9$

A Center $(-2, 1)$ radius 3

B Center $(-2, 1)$ radius 9

C Center $(1, -2)$ radius 3

D Center $(1, -2)$ radius 9

Compare to $(x-h)^2+(y-k)^2=r^2$

$h=1, \quad k=-2$

$r^2=9 \rightarrow r=3$

center $\rightarrow (1, -2)$ **C**

15.16 Determine the equation of circle with center at the origin.

A $x^2+y^2=61$ **B** $9x^2+16y^2=25$

C $x+y=36$ **D** $(x-9)^2+(y+16)^2=36$

Compare to $(x-h)^2+(y-k)^2=r^2$

$(h, k) \rightarrow (0, 0)$

$(x-0)^2+(y-0)^2=r^2$

$x^2+y^2=61$ **A**

15.17 The circle $(x-2)^2+(y-4)^2=16$ passes through the point …..

A $(8, 2)$ **B** $(2, 1)$

C $(0, 1)$ **D** $(2, 8)$

If the circle passes through a point, then the point should satisfy its equation.

$(x-2)^2+(y-4)^2=16$

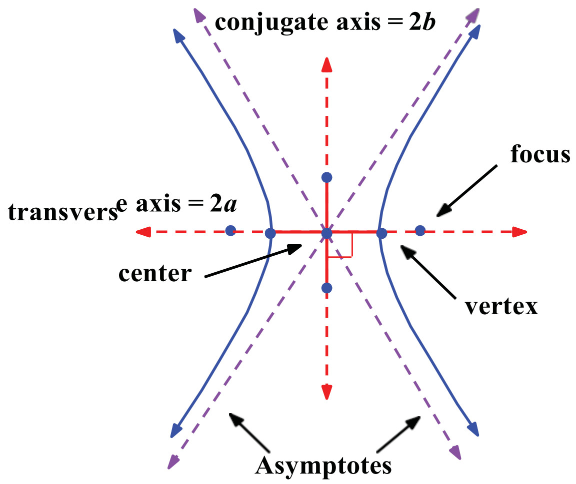
$(2-2)^2+(8-4)^2=15$

$0+4^2=16$ **D**

Hyperbolas

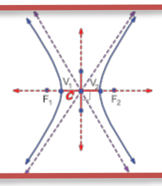
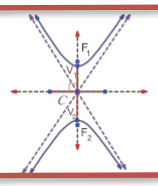
A hyperbola is all the points in a plane such that the absolute value of the difference of the distances from two foci is constant.

Like an ellipse, a hyperbola has two axes of symmetry. The ***transverse*** axis has a length of $2a$ units and connects the vertices. The ***conjugate*** axis is perpendicular to the transverse, passes through the center and has a length of $2b$ units.



Distance and Symbols		
From	To	Symbol
Vertex	Center	a
Vertex	Asymptote	b
Focus	Center	c
Vertex	Vertex	$2a$ = transverse axis
perpendicular to the transverse, passes through the center		$2b$= conjugate axis
Focus	Focus	$2c$

Standard Forms of Equations for Hyperbolas

Equation	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$
Graph		
Orientation	Horizontal transverse axis	Vertical transverse axis
Center	(h, k)	(h, k)
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Foci	$(h \pm c, k)$	$(h, k \pm c)$
Transverse *axis*	$y = k$	$x = h$
Conjugate *axis*	$x = h$	$y = k$
Asymptotes	$(y-k)=\pm\frac{b}{a}(x-h)$	$(y-k)=\pm\frac{a}{b}(x-h)$
a, b, c **relationship**	$c^2=a^2+b^2$ $c=\sqrt{a^2+b^2}$	
Eccentricity	$e=\frac{c}{a}$ $\quad e>1$	

15.19 Find the equation of the asymptotes of the hyperbola $\frac{x^2}{16}-\frac{y^2}{25}=1$

A $y=\pm\frac{4}{5}x$ **B** $y=\pm\frac{25}{16}x$

C $y=\pm\frac{5}{4}x$ **D** $y=\pm\frac{16}{25}x$

The variable y is at the right; therefore, the transverse is horizontal

$b^2=25 \rightarrow b=5$

$a^2=16 \rightarrow a=4$

$(y-k)=\pm\frac{b}{a}(x-h)$

$(y-0)=\pm\frac{5}{4}(x-0)$ $\quad h=0, k=0$

$y=\pm\frac{5}{4}x$ ⪢ **C**

15.18 Find the equation of the transverse *axis* of the hyperbola $\frac{(x-3)^2}{16}-\frac{(y-2)^2}{25}=1$

A $y=-2$ **B** $y=2$

C $x=3$ **D** $x=-3$

The variable y is at the right; therefore, the transverse is horizontal $\boldsymbol{y=k \rightarrow y=2}$

⪢ **B**

15.20 Find the distance from the center to the vertex of the hyperbola $\frac{(x-3)^2}{9}-\frac{(y-3)^2}{25}=1$

A 9 **B** 3

C 25 **D** 5

Distance is $\boldsymbol{a}$

$a^2=9 \rightarrow a=3$ ⪢ **B**

15.21 Find the length of the transverse to the hyperbola $\frac{(x-3)^2}{36}-\frac{y^2}{25}$

A 6 **B** 36

C 12 **D** 5

length of the transverse is $2a \rightarrow a^2=36$

$a=6$

$\rightarrow 2a=12$ ⪢ **C**

15.22 Find the center of the hyperbola $\frac{(y-3)^2}{49}-\frac{(x+5)^2}{36}=1$

A $(-5,-3)$ **B** $(5,-3)$

C $(5, 3)$ **D** $(-5, 3)$

$h=-5, k=3$

Center $(h, k) \rightarrow (-5, 3)$ ⪢ **D**

15.23 Find the intersection point of the transversal and the conjugate *axis* of the hyperbola $\frac{(y+3)^2}{16}-\frac{(x-4)^2}{25}=1$

A $(4, 3)$ **B** $(3, 4)$

C $(-4, 3)$ **D** $(4, -3)$

The intersection point is the same as the center.

$h=4, k=-3$

Center $(h, k) \rightarrow (4, -3)$ ⪢ **D**

15.24 Determine the conic section that has eccentricity greater than 1 $(e > 1)$

A Circle B Parabola
C Ellipse D Hyperbola

Recall Circle $\to e = 0$

Ellipse $\to 0 < e < 1$

Hyperbola $\to e > 1$

D

15.25 Find the equation of the asymptotes of the hyperbola $\frac{(y-5)^2}{16} - \frac{(x+1)^2}{25} = 1$

A $y - 5 = \pm\frac{16}{25}(x+1)$ B $y - 5 = \frac{25}{16}(x+1)$

C $y - 5 = \pm\frac{4}{5}(x+1)$ D $y - 5 = \pm\frac{5}{4}(x+1)$

Since y is the first term; then it has a vertical transverse axis. $k = 5,\ h = -1$

$a^2 = 16$

$a = 4$

$b^2 = 25$

$b = 5$

Asymptotes $y - k = \pm\frac{a}{b}(x - h)$

$y - 5 = \pm\frac{4}{5}(x+1)$ **C**

15.26 Find the eccentricity of hyperbola $\left(\frac{x}{5} - \frac{y}{4}\right)\left(\frac{x}{5} + \frac{y}{4}\right) = 1$

A $\frac{\sqrt{41}}{5}$ B $\frac{\sqrt{41}}{4}$

C $\frac{3}{5}$ D $\frac{3}{4}$

Recall $(x-y)(x+y) = x^2 - y^2$

$\left(\frac{x}{5} - \frac{y}{4}\right)\left(\frac{x}{5} + \frac{y}{4}\right) = \frac{x^2}{5^2} - \frac{y^2}{4^2} = 1$

$a = 5$

$b = 4$

$c = \sqrt{a^2 + b^2}$

$c = \sqrt{5^2 + 4^2} = \sqrt{41}$

$e = \frac{c}{a}$

$= \frac{\sqrt{41}}{5}$ **A**

5) Identify Conic Sections

You can determine the type conic when the equation for the conic in general form.

$Ax^2 + Bxy + Cy^2 + Dx + Ey\ F = 0$ The discriminant, $B^2 - 4AC$, can be used to identify the conic.

Discriminant	Conic
$B^2 - 4AC < 0$, $B = 0$ and $A = C$	Circle
$B^2 - 4AC < 0$, either $B \neq 0$ or $A \neq C$	Ellipse
$B^2 - 4AC = 0$	Parabola
$B^2 - 4AC > 0$	Hyperbola

15.27 Identify the conic section $4x^2 + 3y^2 - 2x + 5y - 60 = 0$

A Circle B Ellipse
C Parabola D Hyperbola

$A = 4,\ B = 0$ and $C = 3$

$B^2 - 4AC = 0^2 - 4(4)(3)$

$= -48 < 0$

$A \neq C \to$ Ellipse **B**

15.28 Identify the conic section $18x - 12y^2 + 4xy + 10x^2 - 6y + 24 = 0$

A Circle B Ellipse
C Parabola D Hyperbola

$A = 10,\ B = 4$ and $C = -12$

$B^2 - 4AC = 4^2 - 4(10)(-12) = 496 > 0 \to$ Hyperbola

D

15.29 Find c such that the conic equation forms a circle $x^2 + Cy^2 + 4x - 5y - 27 = 0$

A 27 B -5
C 4 D 1

$A = 1,\ B = 0$ and $C = ?$

In a circle $A = C$

therefore, $C = A = 1$ **D**