CHAPTER (2)

MATRICES

The number of rows m and columns n is called the order of them matrix, and denoted by $m \times n$. Each element of a matrix is often denoted by a variable with two subscripts. For instance a_{23} represents the element at the second row and the third column of the matrix.

Example

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 7 \\ 0 & 1 \end{bmatrix}$$ the order of the matrix A Is

3×2 and $a_{32} = 1$

1. What is the order of the matrix?

$$A = \begin{bmatrix} -3 & 1 & 0 & 1 & 4 \\ 2 & -1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 \end{bmatrix}$$

A 3×5 **B** 5×3

C 3×3 **D** 5×5

The number of rows m is 3 and the number of columns n is 5, therefore $m \times n$ is 3×5

➤ A

2. Find the value of the element a_{31}

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \\ -1 & 5 & -2 \end{bmatrix}$$

A 0 **B** 2

C -1 **D** -2

a_{31} is the element in third row and first column $\to -1$

➤ C

Operations on Matrices

1- Equal Matrices

Two matrices are equal if:

1- They have the same dimensions $m \times n$

2- Every element a_{ij} of the first matrix is equal to its corresponding element b_{ij} in the second matrix.

Example

$a = [2,1,5] \quad B = [2,5,1]$

Although both matrices are of the order 1×3 but a_{12} of $A \neq b_{12}$ of B therefore $A \neq B$

Example

Find the value of x and y if $\begin{bmatrix} 3 & x-4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 14 \\ 1 & y+1 \end{bmatrix}$

Solution:

Since the two matrices are equal then

$$a_{ij} = b_{ij} \to x - 4 = 14 \quad \text{and} \quad y + 1 = 3$$
$$x = 18 \qquad y = 2$$

2- Addition and Subtraction

The addition or subtraction of two matrices A and B which must be of the same order is carried out by adding or subtracting the corresponding elements of the two matrices.

Example

Find C if $C = A + B$ and $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix}$

Solution

$$C = A + B = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 3 & 9 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 3+1 \\ 1+3 & -1+9 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 \\ 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

3. Evaluate

A $\begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

C $\begin{bmatrix} -9 & -1 \\ 0 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3+(-3) & -1+1 \\ 0+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

➤ B

3- Scalar Multiplication

To obtain the product of a scalar and a matrix, each element of the matrix must be multiplied by the scalar.

Example

$$A=\begin{bmatrix}7 & 3\\ -1 & 4\end{bmatrix} \qquad c=2$$

$$cA=\begin{bmatrix}2\times7 & 2\times3\\ 2(-1) & 2\times4\end{bmatrix}=\begin{bmatrix}14 & 6\\ -2 & 8\end{bmatrix}$$

4. **If** $\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}+2\begin{bmatrix}-1 & 4\\ 3 & 10\end{bmatrix}=\begin{bmatrix}0 & 2x+1\\ y-1 & 25\end{bmatrix}$

then find $x-y$

A 5 **B** -5
C 10 **D** -10

$$\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}+2\begin{bmatrix}-1 & 4\\ 3 & 10\end{bmatrix}=\begin{bmatrix}0 & 2x+1\\ y-1 & 25\end{bmatrix}$$

$$\begin{bmatrix}2 & 3\\ 3 & 5\end{bmatrix}+\begin{bmatrix}-2 & 8\\ 6 & 20\end{bmatrix}=\begin{bmatrix}0 & 2x+1\\ y-1 & 25\end{bmatrix}$$

$$\begin{bmatrix}0 & 11\\ 9 & 25\end{bmatrix}=\begin{bmatrix}0 & 2x+1\\ y-1 & 25\end{bmatrix}$$

$$a_{12}=b_{12}\longrightarrow 11=2x+1$$
$$10=2x$$
$$5=x$$
$$a_{21}=b_{21} \qquad 9=y-1$$
$$10=y$$
$$x-y=5-10$$
$$=-5$$

➢ **B**

5. **Evaluate** $2\begin{bmatrix}3 & 4\\ 1 & 0\end{bmatrix}-3\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$

A $\begin{bmatrix}2 & 5\\ 1 & -1\end{bmatrix}$ **B** $\begin{bmatrix}9 & 7\\ 2 & 1\end{bmatrix}$

C $\begin{bmatrix}3 & 11\\ 2 & -3\end{bmatrix}$ **D** $\begin{bmatrix}2 & 3\\ 1 & -1\end{bmatrix}$

$$2\begin{bmatrix}3 & 4\\ 1 & 0\end{bmatrix}-3\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$$

$$=2\begin{bmatrix}3 & 4\\ 1 & 0\end{bmatrix}+(-3)\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$$

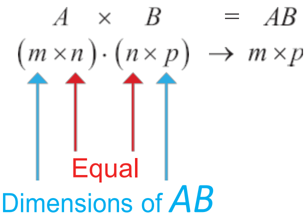
$$=\begin{bmatrix}2\cdot3 & 2\cdot4\\ 2\cdot1 & 2\cdot0\end{bmatrix}+\begin{bmatrix}-3\cdot1 & -3\cdot-1\\ -3\cdot0 & -3\cdot1\end{bmatrix}$$

$$=\begin{bmatrix}6 & 8\\ 2 & 0\end{bmatrix}+\begin{bmatrix}-3 & 3\\ 0 & -3\end{bmatrix}=\begin{bmatrix}3 & 11\\ 2 & -3\end{bmatrix}$$

➢ **C**

4- Matrix Multiplication

You can multiply two matrices A and B only if the number of columns in A is equal to the number of rows in B



Example

Defined multiplication

$A(2\times3) \qquad B(3\times4)$ (3 = 3)

AB dimension (2×4)

undefined multiplication

$A(2\times3) \qquad B(2\times4)$ (3 ≠ 2)

AB is undefined

Example

Find AB if

$$A=\begin{bmatrix}0 & 1 & 2\\ 3 & 4 & 5\end{bmatrix} \qquad B=\begin{bmatrix}6 & 7\\ 8 & 9\\ 10 & 11\end{bmatrix}$$

Step 1: Check if the AB is defined or not and

$A(2\times3) \qquad B(3\times2)$ (3 = 3)

AB dimension (2×2)

$$AB=\begin{bmatrix}(0\cdot6)+(1\cdot8)+(2\cdot10) & (0\cdot7)+(1\cdot9)+(2\cdot11)\\ (3\cdot6)+(4\cdot8)+(5\cdot10) & (3\cdot7)+(4\cdot9)+(5\cdot11)\end{bmatrix}$$

$$=\begin{bmatrix}28 & 31\\ 100 & 112\end{bmatrix}$$

Determinants and Inverse

Square matrices have determinants, which are useful in other matrix operations.

For a second-order square matrix

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ determinant $|A|$ is $ad-bc$

Inverse of a matrix is denoted by A^{-1}

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, $A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

If the $|A|=0$ then the matrix does not have inverse.

Example

Does the matrix $\begin{bmatrix} 4 & 12 \\ 3 & 9 \end{bmatrix}$ **have inverse?**

Solution: Find the determinant

$$\begin{aligned}|A| &= 4\cdot 9-12\cdot 3 \\ &= 36-36 \\ &= 0\end{aligned}$$

Since the determinant is zero and $\frac{1}{|A|}$ is undefined then the matrix doesn't have inverse.

6. Find the inverse of the matrix $\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

A $\begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix}$

B $\frac{1}{7}\begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix}$

C $7\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$

D $\frac{1}{7}\begin{bmatrix} -3 & 1 \\ 5 & 4 \end{bmatrix}$

Step 1

Find the determinant $|A|=3\cdot 4-5\cdot 1 = 12-5 = 7$

Step 2

Find the inverse $A^{-1}=\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\frac{1}{7}\begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix}$

➢ **B**

7. Find x such that the matrix $\begin{bmatrix} 8 & -2 \\ x & x+1 \end{bmatrix}$ **does not have inverse**

A $\frac{4}{5}$

B $\frac{-4}{5}$

C $\frac{5}{4}$

D $\frac{-5}{4}$

If the matrix does not have inverse the determinant should equal zero

$$\begin{aligned}|A| &= 8(x+1)-(-2)(x)=0 \\ &= 8x+8+2x=0 \\ &= 10x+8=0 \rightarrow\end{aligned}$$

$$x=\frac{-8}{10}=\frac{-4}{5}$$

➢ **B**

* For a third-order square matrix use the diagonal method. Rewrite the first two columns on the right of the matrix

$$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$$|A|=(aei+bfg+cdh)-(gec+hfa+idb)$$

Example

Find the determinant of $\begin{bmatrix} -7 & -10 & 4 \\ 3 & -9 & 2 \\ 7 & 1 & 2 \end{bmatrix}$

Solution: $\begin{bmatrix} -7 & -10 & 4 \\ 3 & -9 & 2 \\ 7 & 1 & 2 \end{bmatrix}\begin{matrix} -7 & -10 \\ 3 & -9 \\ 7 & 1 \end{matrix}$

$$(126-140+12)-(-252-14-60)=324$$

The determinant of 3×3 matrix help to find the area of a triangle

Example

Find the area of the triangle if it's vertices at

$A(0,0)$, $B(-2,8)$, $C(4,12)$

Step 1

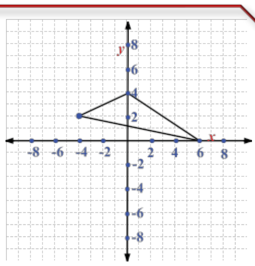
Find the determinant

$$\begin{bmatrix} 0 & 0 & 1 \\ -2 & 8 & 1 \\ 4 & 12 & 1 \end{bmatrix}\begin{matrix} 0 & 0 \\ -2 & 8 \\ 4 & 12 \end{matrix}$$

$$|A|=|(0+0-24)-(32+0+0)|=|-56|=56$$

$$\begin{aligned}\text{Area} &= \frac{1}{2}|A| \\ &= \frac{1}{2}\cdot 56 \\ &= 28\end{aligned}$$

8. Find the area of the triangle



A -14

B 14

C 28

D -28

$$|A|=\begin{vmatrix} -4 & 2 & 1 \\ 6 & 0 & 1 \\ 0 & 4 & 1 \end{vmatrix}\begin{matrix} -4 & 2 \\ 6 & 0 \\ 0 & 4 \end{matrix}$$

$$\begin{aligned}&=|(0+0+24)-(0-16+12)| \\ &=|24+4| \\ &=28\end{aligned}$$

$$\begin{aligned}Area &= \frac{1}{2}|A| \\ &= \frac{1}{2}|28|=14\end{aligned}$$

➢ **B**